Math 206B Lecture 1 Notes

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1 Teaser of Course Topics

This lecture will be an advertisement for the topics in the course.[1]

1.1 Combinatorics of S_n and applications

We denote $[n] := \{1, \dots, n\}$.

Definition 1.1. The **symmetric group** S_n is the group of bijections $\sigma : [n] \to [n]$.

We can write permutations as products of cycles.

Example 1.1. The permutation $\sigma = \begin{pmatrix} 5 & 1 & 2 & 4 & 3 \end{pmatrix}$ represents the bijection sending $1 \mapsto 5$, $2 \mapsto 1$, etc.

The conjugacy classes of S_n are the different cycle types. These correspond to partitions of n. Let $p(n)$ be the number of conjugacy classes of S_n. Euler showed that

$$1 + \sum_{n=1}^{\infty} p(n) t^n = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}.$$

What does this all have to do with the symmetric group itself? Here is Percy MacMahon's version of the story. If we have a partition, we can think of it as a sequence of numbers, padded with zeros at the end to make it infinite. We can write

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1 - t^i}.$$

If we write the partition over and over in a grid (chopping off an element from the front each time, we can get what is called a **plane partition**. This gives us

$$\sum_{A \in \mathcal{PP}} t^{|A|} = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^i},$$

[1] This is an advertisement of an advertisement, much like the ad before watching a trailer online.

where $|A| = \sum_{i,j} a_{i,j}$ is the sum of the numbers in the plane partition. Why should this be true? This is actually related to the representation theory of S_n.

MacMahon made a conjecture for higher dimensions:

Theorem 1.1.

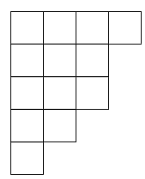
$$\sum_{A \in \mathcal{P}^{(d)}} t^{|A|} = \prod \frac{1}{(1-t^i)^{(i_{d-1})}}.$$

This does not work. It actually fails for $d = 3$ and a low coefficient like the coefficient of t^7. MacMahon did not understand why the formula was true, even though he proved it. Irreducible representations of S_n will correspond to partitions of n $\mathcal{P}_n := \{\lambda \in \mathcal{P} : |\lambda| = n\}$ because these correspond to conjugacy classes of S_n.

Theorem 1.2 (A. Young, 1897)**.** *Let f^λ be the dimension of S^λ. Then f^λ is the number of standard Young tableau with shape λ.*

Definition 1.2. Given a partition λ, the *young diagram* of λ is the partition expressed as stacked rows of boxes.

Example 1.2. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.3. A **Young tableau** is a Young diagram where we fill in the boxes with the numbers 1 to n, according to the rule that the numbers have to be increasing going to the right and going down.

Example 1.3. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

Theorem 1.3 (FRT, c.1960)**.** $f^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}}$, *where $h_{i,j}$ is the length of the hook starting from position i, j and going to the right and downwards.*

From representation theory, we can get that $f^\lambda \mid |S_n|$, so we know that f^λ is $n!$ divided by something. The magic is in what that something is.

1.2 Representation theory of $\mathrm{GL}_n(\mathbb{C})$

A basic representation of $\mathrm{GL}_n(\mathbb{C})$ is ρ_M is the matrix M acting on $\mathbb{C}^n$. Another representation is the determinant map. We will find that representations of $\mathrm{GL}_n(\mathbb{C})$ correspond to sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Theorem 1.4 (Weyl)**.** $\dim(\rho_\lambda)$ *equals the number of semistandard Young tableau with shape* λ*.*

This will be another remarkable product formula. This, paired with the hook-length formula, will pave the way for a nice proof of MacMahon's formula.

1.3 Young graph

Definition 1.4. The **Young graph** is the graph with vertices $\lambda \in \Gamma$, and edges $(\lambda, \mu$ where $\mu \setminus \lambda$ is a single difference.

Essentially, we have taken all young diagrams and made an undirected graph, partially ordering them by containment.

Theorem 1.5. *The number of loops of length* $2n$ *(that do not zigzag up and down) in the Young graph is* $n!$*.*

Proof. We can prove this using basic representation theory.

$$\#\text{ loops} = \sum_{\lambda \in \mathcal{P}_n} (\#\text{ paths } \phi \to \lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} \mathrm{SYT}(\lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2 = |S_n| = n! \qquad \square$$

What about general loops?

Theorem 1.6. *The number of general loops is* $(2n-1)!!$*.*